Recognizing (Unit) Interval Graphs by Zigzag Graph Searches

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SIAM Symposium on Simplicity in Algorithms
January 10, 2021

The Hong Kong Polytechnic University
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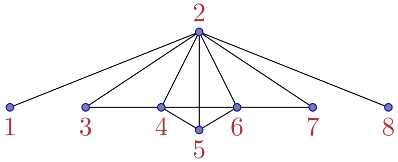
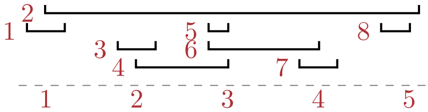
Interval graphs



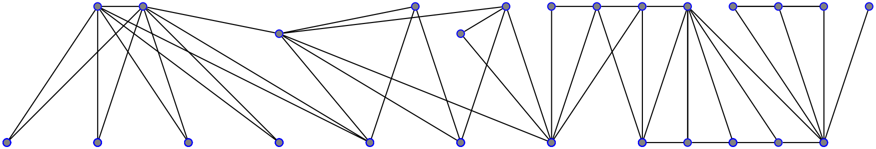
(a) An interval graph.



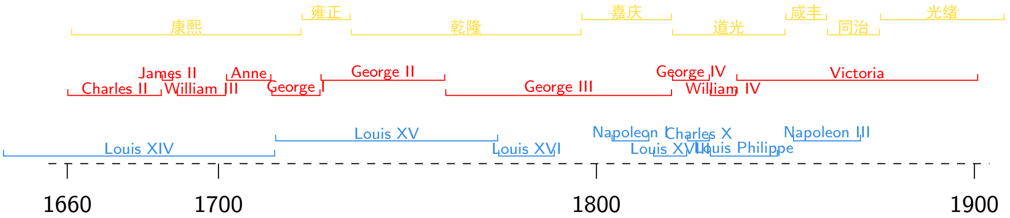
(b) Its interval representation.

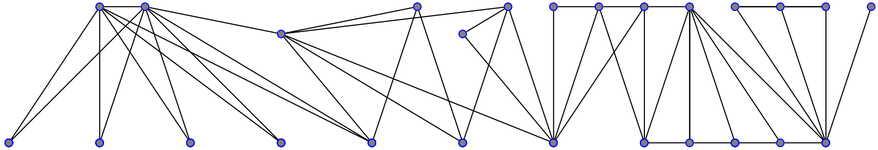
A graph G is an interval graph if there is a set $\mathcal{I}$ of intervals on the real line and a bijection ϕ: $V(G) \to \mathcal{I}$ such that $uv \in E(G) \iff \phi(u) \cap \phi(v) \neq \emptyset$.

Is this an interval graph?

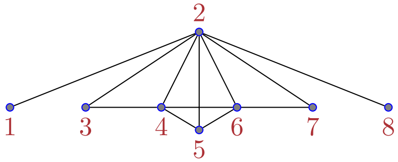


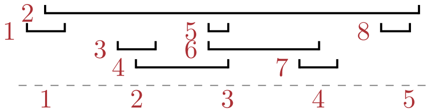
The answer: monarchy rulers of China, France, and UK, 1661–1900





A closer look at interval models



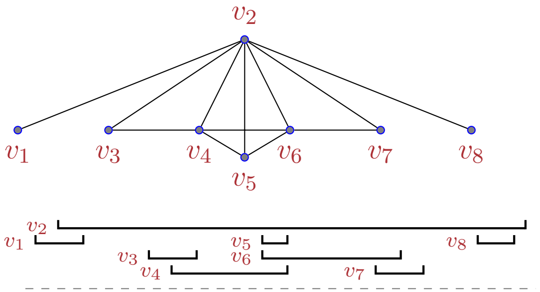


- Each interval representation defines a unique interval graph.
- But an interval graph can have many interval representations.

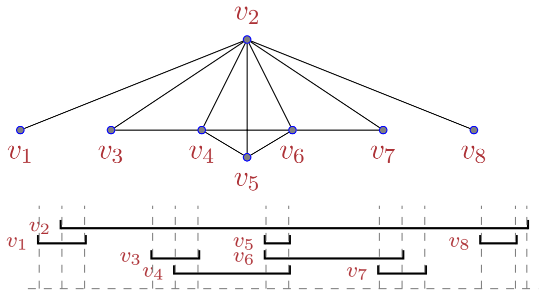
$$I(v) = [\mathrm{lp}(v), \mathrm{rp}(v)]$$

- What're important aren't the absolute positions of the $2n$ endpoints.
- but the ordering, not unique but finite.

Characterization of interval graphs: Clique paths



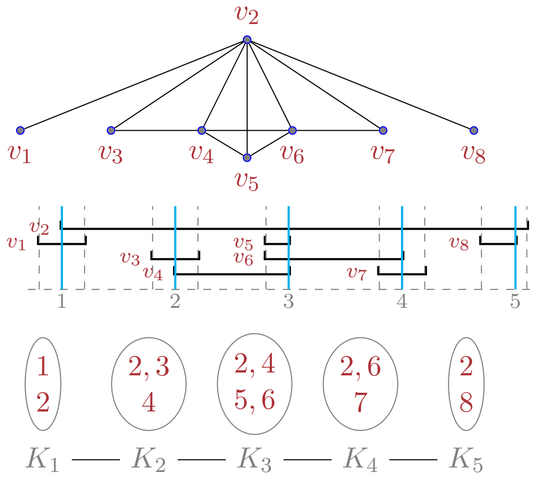
Characterization of interval graphs: Clique paths



For each endpoint, draw a vertical line.
Intervals crossing each line induce a clique.

They include all maximal cliques.

Characterization of interval graphs: Clique paths



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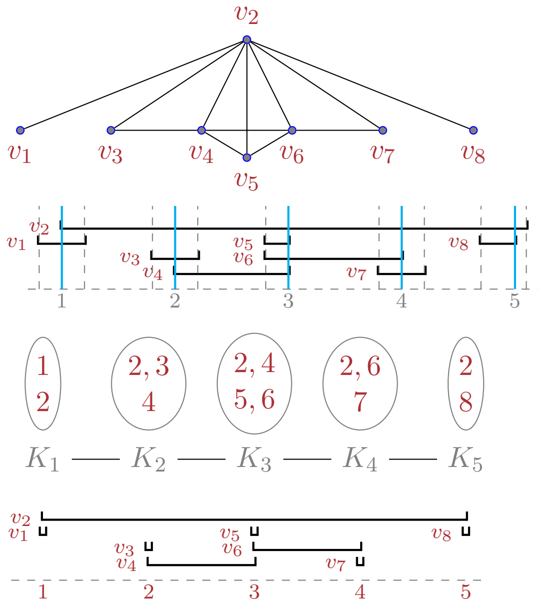
Clique path (a special kind of clique tree):
each node for a maximal clique and
cliques containing v induce a sub-path.

Characterization of interval graphs: Clique paths



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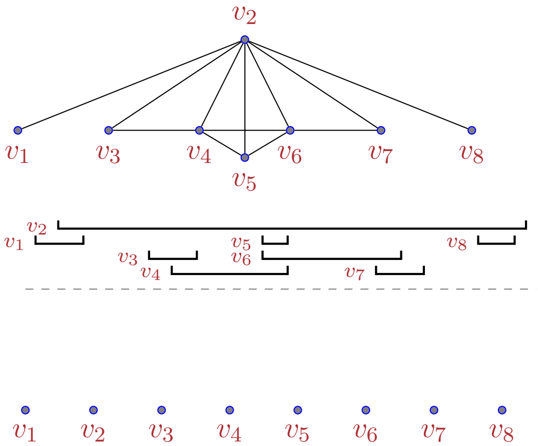
Intervals crossing each line induce a clique.

They include all maximal cliques.

Clique path (a special kind of clique tree):
each node for a maximal clique and
cliques containing v induce a sub-path.

From a clique path we can “read”
$\mathrm{lp}(v) = \min\{i \mid v \in K_i\}$
$\mathrm{rp}(v) = \max\{i \mid v \in K_i\}$

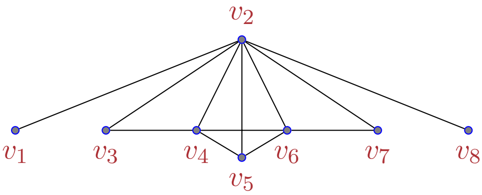
Characterization of interval graphs: Interval ordering

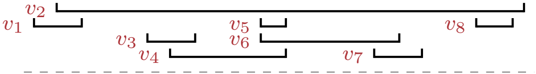


Sort the intervals by their left endpoints. breaking ties arbitrarily.

$i < j < k$ and $v_i \sim v_k$ then $v_i \sim v_j$.

Characterization of interval graphs: Interval ordering





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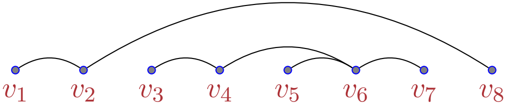
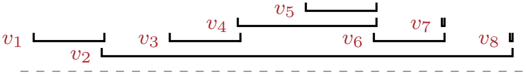
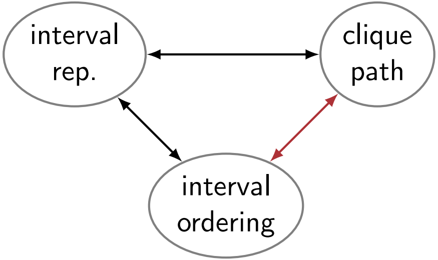


$$\begin{array}{rcccccccc} \sigma: & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8. \\ r(i): & 2, & 8, & 4, & 6, & 6, & 7, & 7, & 8. \end{array}$$



From an ordering we can read:
$I(v_i) = [\sigma(v_i), \sigma(v_{r(i)})]$

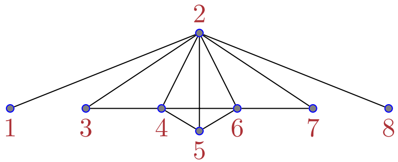
Characterizations of interval graphs (summary)

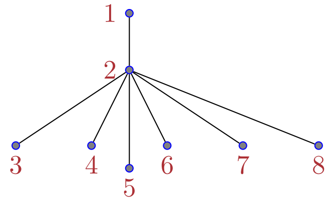


The following are equivalent:

1. G is an interval graph.
2. G has an interval ordering.
3. G has a clique path.

Recursive breadth-first search (BFS)





The output of a BFS is a BFS tree
in which the order of siblings is immaterial.

$3, \ldots, 8$ can be visited in any order.

Since we aim for an ordering.
We may recursively conduct BFS.
$1, 2, [3, \ldots, 8]$
$1, 2, 3, 4, [5, 6], 7, 8$
$1, 2, 3, 4, 5, 6, 7, 8$

Lexicographical breadth-first search (LBFS)

$L_1, L_2 \subseteq \{1, 2, \ldots, n\}$, and $L_1 \neq L_2$.
L_1 is *lexicographically larger* than L_2 if the minimum element in $(L_1 \setminus L_2) \cup (L_2 \setminus L_1)$ belongs to L_1. e.g., $\{1, 2\} > \{1, 3, 4\} > \{1\}$.

---

1. **for each** $v \in V(G)$ **do** $\mathrm{label}(v) \leftarrow \emptyset$;
2. **for** $i = 1, \ldots, n$ **do**
   - 2.1. $S \leftarrow$ unvisited vertices with the lexicographically largest label;
   - 2.2. $v \leftarrow$ any vertex in S;
   - 2.3. $\sigma(v) \leftarrow i$;
   - 2.4. **for each** unvisited neighbor of v **do** add i to $\mathrm{label}(v)$;
3. **return** σ.

Fact: LBFS is the recursive BFS.

Theorem (**rose-76-vertex-elimination rose-76-vertex-elimination**).
On a chordal ($\{C_\ell \mid \ell \geq 4\}$-free) graph, the last vertex of an LBFS is simplicial.

Interval orderings and LBFS orderings

Proposition. Any interval ordering σ of an interval graph G is an LBFS ordering of G.

Proof. We may assume without loss of generality that $\sigma(v_i) = i$ for all $i = 1, \ldots, n$. For any i, p, and q with $i < p < q$, if $v_i v_q \in E(G)$, then $v_i v_p \in E(G)$ as well. Thus,

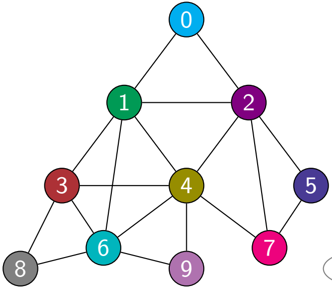
$$\{v_1, \ldots, v_{p-1}\} \cap N(v_q) \subseteq \{v_1, \ldots, v_{p-1}\} \cap N(v_p),$$

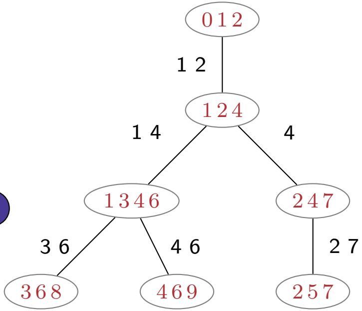
and after visiting $\{v_1, \ldots, v_{p-1}\}$, the label of v_p is no smaller than that of v_q. $\square$

— The other direction is not true: most LBFS orderings are not interval orderings.

Theorem (corneil-10-end-vertices-lbfs corneil-10-end-vertices-lbfs).
If G is an interval graph, then an interval ordering of G can be produced by less than n sweeps of LBFS$^+$ (a variant of LBFS).

Maximal cliques of a chordal graph





These representations are not unique
The clique tree implies a representation.
Two subtrees intersect if they have one point in common.

Recognition algorithms for interval graphs

Build a clique tree and check whether it can be transformed into a clique path:

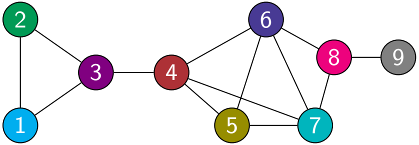
- **booth-76-pq-tree booth-76-pq-tree** PQ–trees.
- **korte-89-recognizing-interval-graphs korte-89-recognizing-interval-graphs** LBFS and Modified PQ–trees.
- Hsu and McConnell **hsu-03-pc-trees**
- Hsu and Ma **hsu-99-recognizing-interval-graphs** modular decomposition
- **habib-00-LBFS-and-partition-refinement habib-00-LBFS-and-partition-refinement** LBFS and partition refinement.

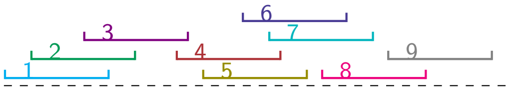
Construct an interval ordering:

- **simon-91-interval simon-91-interval** multiple sweeps LBFS
- **corneil-09-lbfs-strucuture-and-interval-recognition corneil-09-lbfs-strucuture-and-interval-recognition** 6 sweeps of LBFS
- **li-14-lbfs-interval-recognition li-14-lbfs-interval-recognition** 4 sweeps of LBFS

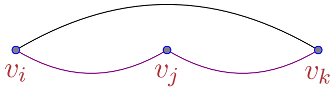
Appetizer: Unit Interval Graphs

Unit/proper interval graphs





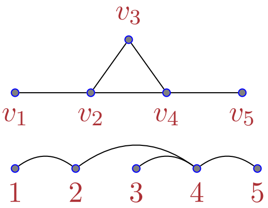
unit: All the intervals have the same length.
proper: No interval properly contains another.



Umbrella (proper interval) ordering:
$v_i \sim v_k \Rightarrow v_i \sim v_j$ and $v_j \sim v_k$

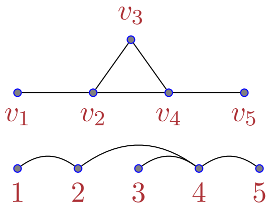
- Sorting vertices in the order of their left endpoints in a proper representations.
- The reversal of an umbrella ordering is also an umbrella ordering.

Umbrella orderings $\to$ clique paths and representations



$v_{r(i)}$: last neighbor of v_i in σ

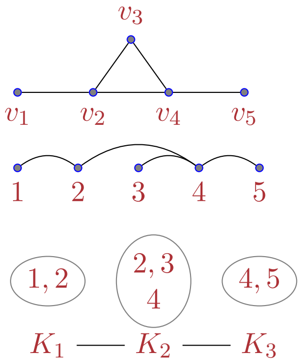
Umbrella orderings $\to$ clique paths and representations



$v_{r(i)}$: last neighbor of v_i in σ

n cliques: $K_i = \{v_i, \ldots, v_{r(i)}\}$
$\{v_1, v_2\}$; $\{v_2, v_3, v_4\}$; $\{v_3, v_4\}$; $\{v_4, v_5\}$; $\{v_5\}$

Umbrella orderings $\to$ clique paths and representations



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n cliques: $K_i = \{v_i, \dots, v_{r(i)}\}$

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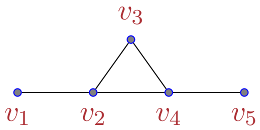
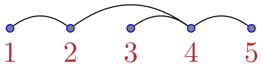
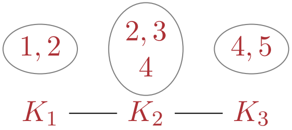
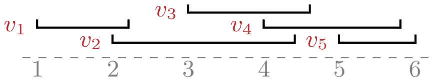
Umbrella orderings $\to$ clique paths and representations









$v_{r(i)}$: last neighbor of v_i in σ

n cliques: $K_i = \{v_i, \dots, v_{r(i)}\}$

$\{v_1, v_2\}$; $\{v_2, v_3, v_4\}$; $\underline{\{v_3, v_4\}}$; $\{v_4, v_5\}$; $\underline{\{v_5\}}$

$$I(v_i) = \left[\sigma(v_i), \sigma(v_{r(i)}) + \frac{\sigma(v_i)}{n}\right]$$

$$I(v_1) = \left[\sigma(v_1), \sigma(v_2) + \frac{\sigma(v_1)}{5}\right] = [1, 2.2];$$

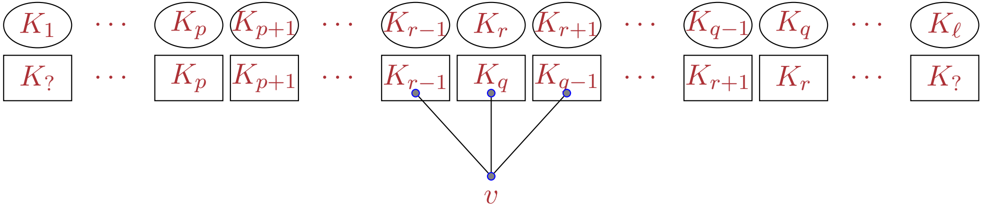
$$I(v_2) = \left[\sigma(v_2), \sigma(v_4) + \frac{\sigma(v_2)}{5}\right] = [2, 4.4];$$

$I(v_3) = [3, 4.6]$; $\quad I(v_4) = [4, 5.8]$; $\quad I(v_5) = [5, 6]$

Uniqueness of clique paths

Proposition. A connected proper interval graph has a unique clique path.

Proof 1: Suppose that there are two clique paths. We can find a minimal sub-path of the first that do not appear in the same order in the second.

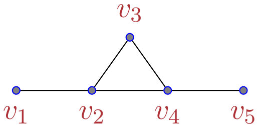


2. Follows from Hsu's theorem if all its modules are cliques. Trivial if $K_r \vee (K_p + K_q)$. Otherwise, removing universal vertices leaves another connected proper interval graph.
3. Let $K_1, \ldots, K_\ell$ be a clique path. A simplicial vertex in $K_i, 1 < i < \ell$ is the nose of a bull. So the ends of any clique path of G must be K_1 and K_ℓ. Then K_2 and $K_{\ell-1}$.

Theorem (Deng, Hell, Huang 1996).
If a unit interval graph contains no true twins, then it has a unique ordering up to ...

LBFS on unit interval graphs

We look for an umbrella ordering, so should start from "an end" of the graph.

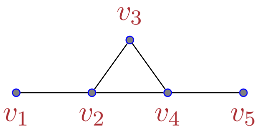


From v_1 may give $\langle v_1, v_2, v_4, v_3, v_5 \rangle$
or $\langle v_1, v_2, v_3, v_4, v_5 \rangle$

It cannot distinguish v_3 and v_4.

LBFS on unit interval graphs

We look for an umbrella ordering, so should start from "an end" of the graph.



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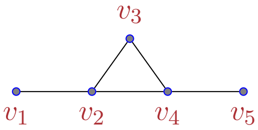
It cannot distinguish v_3 and v_4.

From v_5 may give $\langle v_5, v_4, v_3, v_2, v_1 \rangle$
or $\langle v_5, v_4, v_2, v_3, v_1 \rangle$

It cannot distinguish v_2 and v_3.

LBFS on unit interval graphs

We look for an umbrella ordering, so should start from "an end" of the graph.



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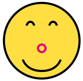
From v_5 may give $\langle v_5, v_4, v_3, v_2, v_1 \rangle$
or $\langle v_5, v_4, v_2, v_3, v_1 \rangle$

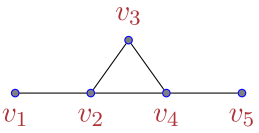
It cannot distinguish v_2 and v_3.

Together they can:
v_2 before v_3; v_4 after v_3.



LBFS on unit interval graphs

We look for an umbrella ordering, so should start from "an end" of the graph.



From v_1 may give $\langle v_1, v_2, v_4, v_3, v_5 \rangle$
or $\langle v_1, v_2, v_3, v_4, v_5 \rangle$

It cannot distinguish v_3 and v_4.

From v_5 may give $\langle v_5, v_4, v_3, v_2, v_1 \rangle$
or $\langle v_5, v_4, v_2, v_3, v_1 \rangle$

It cannot distinguish v_2 and v_3.

Not done yet

1. Where are "the ends"?
2. How to combine the two orderings?
3. Whether the output is correct?

Together they can:
v_2 before v_3; v_4 after v_3.

The ends

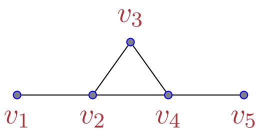
1. find "the ends"?
2. combine complementary orderings
3. verify the output

A *terminal*: a simplicial vertex in the first clique of a clique path.
More restrictive than end interval (Gimbel **gimbel-88-end-vertices**)

Proposition. The last vertex of an LBFS ordering is a terminal of G.
Corollary. An LBFS starting from a terminal ends with another terminal.

LBFS$^+$

1. find "the ends"?
2. combine complementary orderings
3. verify the output



$\sigma : v_1, v_2, v_4, v_3, v_5$

$\sigma^+ : v_5, v_4, v_3, v_2, v_1$

Input: A graph G, and an LBFS ordering σ of G.

1. **for each** $v \in V(G)$ **do** $\mathrm{label}(v) \leftarrow \emptyset$;
2. **for** $i = 1, \ldots, n$ **do**

2.1. $S \leftarrow$ unvisited vertices with the lexicographically largest label;

2.2. $v \leftarrow$ the last vertex of $\sigma|_S$;

2.3. $\sigma^+(v) \leftarrow i$;

2.4. **for each** unvisited neighbor of v **do** add i to $\mathrm{label}(v)$;

3. **return** σ^+.

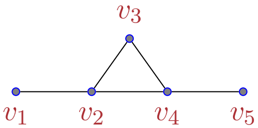
The algorithm for recognizing unit interval graphs

1. $\tau \leftarrow$ an LBFS ordering of G;
2. $\sigma \leftarrow \text{LBFS}^+(G, \tau)$;
3. $\sigma^+ \leftarrow \text{LBFS}^+(G, \sigma)$;
4. **if** σ^+ is an umbrella ordering of G **then return** "yes";
5. **else return** "no."

Theorem (Corneil 2004).
If G is a proper interval graph, then three sweeps of LBFS produce an umbrella ordering.

Whether the output is correct?

1. find "the ends"?
2. combine complementary orderings
3. verify the output



Final output σ^+:

v_1, v_2, v_3, v_4, v_5

Order $N(v)$ in the reversed order of σ^+

$v_1 : v_2$

$v_2 : v_4 \to v_3 \to v_1$

$v_3 : v_4 \to v_2$

$v_4 : v_5 \to v_3 \to v_2$

$v_5 : v_4$

check whether the list for v_i starts from
$r(i), r(i) - 1, \ldots, i + 1$

Do the same operations for the reversal of σ^+
return "yes" if both tests are passed.

A two-sweep algorithm

Input: A connected graph G and a vertex $s \in V(G)$.

1. **for each** $v \in V(G)$ **do** $\text{label}(v) \leftarrow \emptyset$;
2. $\sigma(s) \leftarrow 1$;
3. **for** $i = 2, \ldots, n$ **do**

   3.1. $S \leftarrow$ unvisited vertices with the lexicographically largest label;

   3.2. $v \leftarrow$ a vertex with the minimum degree in S;

   3.3. $\sigma(v) \leftarrow i$;

   3.4. **for each** unvisited neighbor of v **do** add i to $\text{label}(v)$;

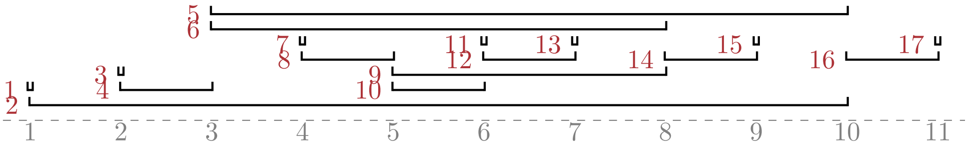
4. **return** σ.

---

1. $u \leftarrow$ an end vertex of G;
2. $\sigma \leftarrow \text{LBFS}^{\delta}(G, u)$;
4. **if** σ is an umbrella ordering of G **then return** "yes";
5. **else return** "no."

Recognition of Interval Graphs

Why interval graphs are difficult?



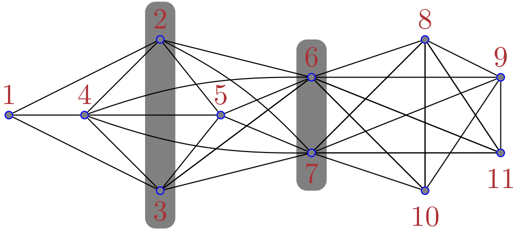
- for $1 \leq i < j < k \leq n$, $v_i \sim v_k$ implies only $v_i \sim v_j$.
- the reversal of an interval ordering may not be an interval ordering. $1, 2, \ldots, 17$ is an interval ordering, but $17, 16, \ldots, 1$ is not.
- clique paths may not be unique.
- An LBFS does not necessarily go from left to right, it can jump arbitrarily (zigzag). e.g., $1, 2, 16, 11, \ldots$

An oversimplistic summary

Breaking fake twins (unit interval graphs)[1] and fake modules (interval graphs).

[1] If an unit interval graph contains a non-clique module, it's more or less trivial.

Modules



M is a *module* if for all $u, v \in M$ and $x \notin M$, $xu \in E(G)$ if and only if $xv \in E(G)$.

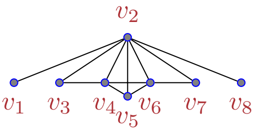
vertices in M have the same neighborhood outside M.

- *trivial modules*: $V(G)$, $\emptyset$, and $\{v\}$. the graph has 13 trivial modules.
- Other simple examples: components and twin classes (clique modules, independent set modules).

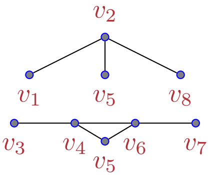
Interval graphs and modules

Let M be a non-clique module of G, and $G' = G - (M \setminus \{v\})$ for some vertex $v \in M$.

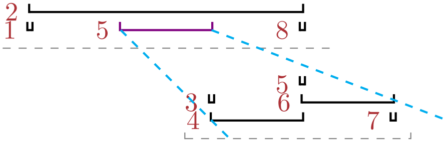
Proposition. G is an interval graph if and only if (1) $N(M)$ is a clique; and (2) both G' and $G[M]$ are interval graphs.



(a) G



(b) G' and $G[M]$

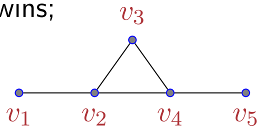


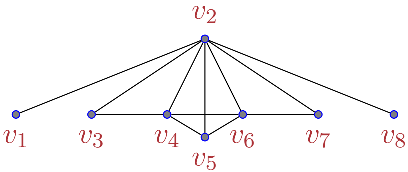
(c) interval representations for G' and $G[M]$.

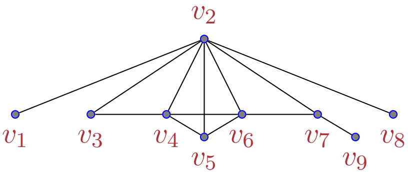
Why it worked and why it doesn't work

For the recognition of proper interval graphs,

- it is neither necessary nor possible to distinguish true twins;
- what we need to do is to distinguish "fake" twins; and
- an LBFS from v_1 cannot distinguish $\{v_3, v_4\}$.

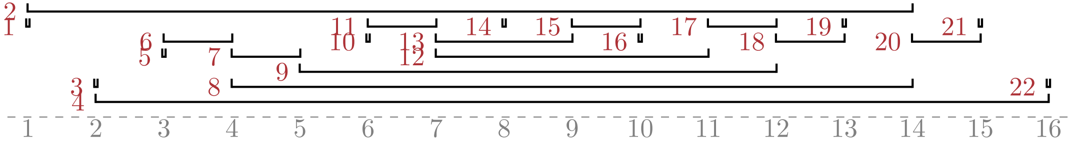






In both graphs, an LBFS from v_1 cannot distinguish $\{v_3, v_4, \ldots, v_8\}$,
a module ($N(M) = N(x) \setminus M$ for all $x \in M$) in the first, but not the second.

Inspiration



$\sigma_1 : 1, 2, 20, 8, 4, [19, 18, 17, 9, 12, 16, 15, 13, 11, 14, 10, 7, 6], 5, 3, 21, 22$

$\sigma_1^+ :$

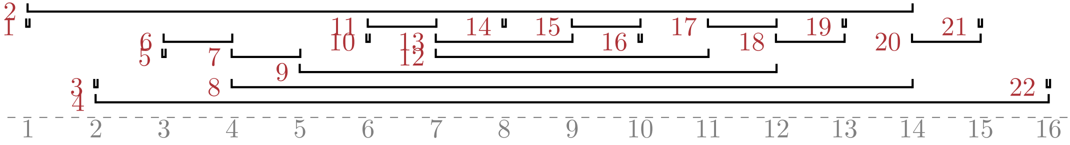
$\sigma_2 : 1, 2, 3, 4, 8, [20, [15, 16, 12, 9, 13, 11, 14, 17, 10, 18, 7, 19, 6]], 5, 21, 22$

$\sigma_2^+ :$

$\sigma_3 : 1, 2, [4, 20, 8, [6, 7, 9, 18, 17, 12, [11, 13, 15, 14, 16], 10, 19], 5, 3], 21, 22$

$\sigma_3^+ :$

Inspiration



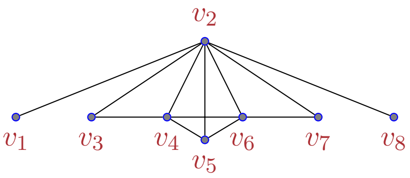
$$\sigma_1 : 1, 2, 20, 8, 4, [19, 18, 17, 9, 12, 16, 15, 13, 11, 14, 10, 7, 6], 5, 3, 21, 22$$
$$\sigma_1^+ : 22, 4, 21, 20, 8, 2, [6, 7, 9, 10, 11, 13, 12, 14, 15, 16, 17, 18, 19], 5, 3, 1$$

$$\sigma_2 : 1, 2, 3, 4, 8, [20, [15, 16, 12, 9, 13, 11, 14, 17, 10, 18, 7, 19, 6]], 5, 21, 22$$
$$\sigma_2^+ : 22, 4, 21, 20, 8, 2, [6, 7, 9, 18, 17, 12, 14, 13, 11, 15, 16, 10, 19], 5, 3, 1$$

$$\sigma_3 : 1, 2, [4, 20, 8, [6, 7, 9, 18, 17, 12, [11, 13, 15, 14, 16], 10, 19], 5, 3], 21, 22$$
$$\sigma_3^+ : 22, 4, 21, 20, 8, 2, [19, 18, 17, 9, 12, [16, 15, 13, 14, 11], 10, 7, 6], 5, 3, 1$$

Snapshot: one-sided module

- Snapshot: the set S in step 2.1 in $\textsc{lbfs}^+$.
- There are n snapshots, the one in the iteration visiting v_i is denoted by $S_\sigma(v_i)$.
- $S_\sigma(v_1) = V(G)$ and $S_\sigma(v_n) = \{v_n\}$.
- All vertices in $S_\sigma(v_i)$ have the label in this iteration, i.e., $\{v_1, \ldots, v_{i-1}\} \cap N(v_i)$.



nontrivial snapshots of $1, \ldots, 8$.

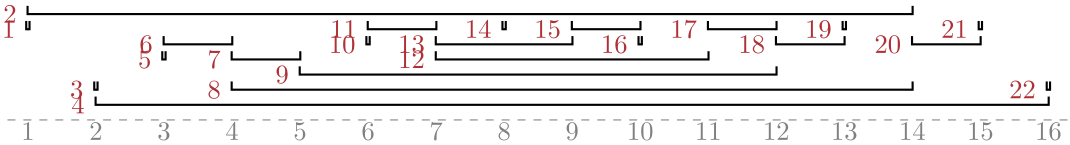
$1 : 1, 2, 3, 4, 5, 6, 7, 8.$

$3 : 3, 4, 5, 6, 7, 8$

$5 : 5, 6$

in summary, $1, 2, [3, 4, [5, 6], 7, 8]$.

Well-anchored orderings



$\sigma_3 : 1, 2, [4, 20, 8, [6, 7, 9, 18, 17, 12, [11, 13, 15, 14, 16], 10, 19], 5, 3], 21, 22$

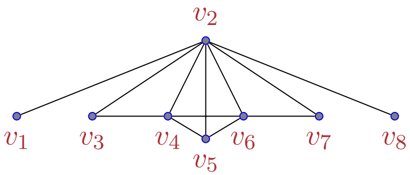
$\sigma_3^+ : 22, 4, 21, 20, 8, 2, [19, 18, 17, 9, 12, [16, 15, 13, 14, 11], 10, 7, 6], 5, 3, 1$

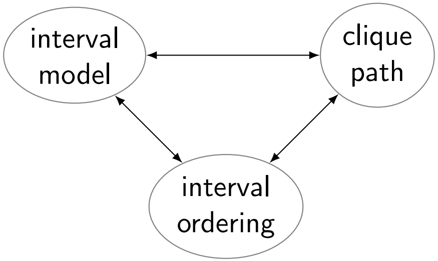
An LBFS ordering σ is *well-anchored* if for each snapshot S, $\sigma|_S$ starts from

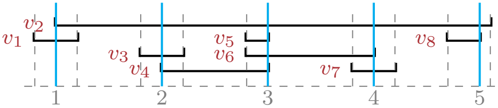
- an exposed vertex (has a neighbor after S); or
- an end vertex of $G[S]$ if no exposed vertex.

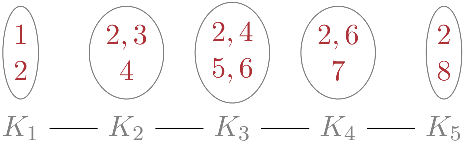
For the snapshot $S = \{6, 7, 9, 10, \ldots, 19\}$ in σ_3, 6 is an exposed vertex

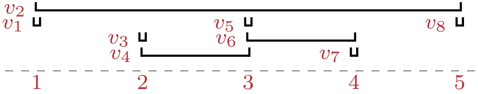
Characterizations of interval graphs (recalled)

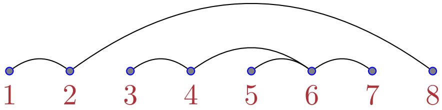




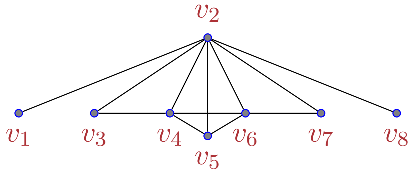


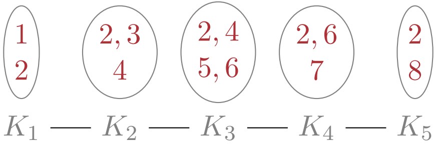






Interval ordering $\to$ clique path



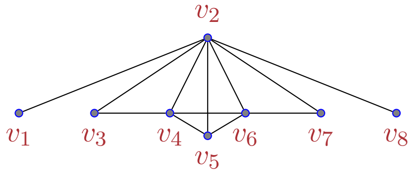


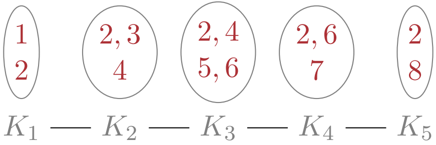
path to ordering:

$K_1, K_2 \setminus K_1, \ldots, K_\ell \setminus K_{\ell-1}$

$1, 2, 3, 4, 5, 6, 7, 8$

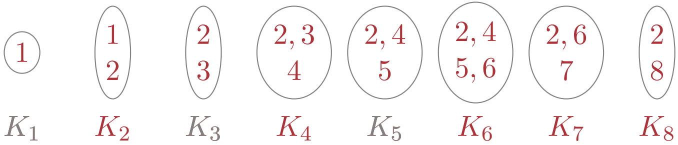
Interval ordering $\to$ clique path





ordering to path:
$S_j = \{v_1 \ldots, v_j\} \cap N[v_j]$;
keep S_j only when $|S_j| \geq |S_{j+1}|$.

path to ordering:
$K_1, K_2 \setminus K_1, \ldots, K_\ell \setminus K_{\ell-1}$



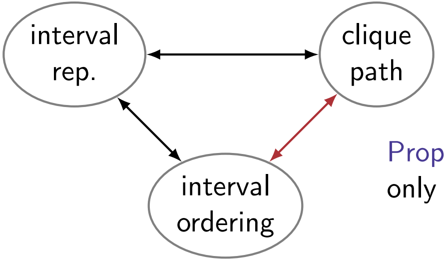
$1, 2, 3, 4, 5, 6, 7, 8$

Interval orderings $\longleftrightarrow$ clique paths

An ordering σ is *consistent with* a clique path $K_1, \ldots, K_\ell$ if σ can be represented as

$$\langle K_1, K_2 \setminus K_1, \ldots, K_\ell \setminus K_{\ell-1} \rangle,$$

where vertices in each set are in any order.



Proposition. An ordering σ is an interval ordering if and only if σ is consistent with some clique path $K_1, \ldots, K_\ell$.

Clique paths and modules

Lemma (C 2021). Let $K_1, K_2, \ldots, K_\ell$ and $K_{b(1)}, K_{b(2)}, \ldots, K_{b(\ell)}$ be two different clique paths of G. If there are p and q, $1 \leq p < q < \ell$ such that $b(p) = 1$ and $b(q) = \ell$, then $\bigcup_{j=b(\ell)}^{\ell} K_j \cup \bigcup_{i=q}^{\ell} K_{b(i)} \setminus (K_{b(\ell)} \cap K_\ell)$ is a nontrivial non-clique module of G.

K_1	$\cdots$	K_p	$\cdots$	K_q	K_{q+1}	$\cdots$	K_ℓ
$K_{b(1)}$	$\cdots$	$K_{b(p)}$	$\cdots$	$K_{b(q)}$	$K_{b(q+1)}$	$\cdots$	$K_{b(\ell)}$

Proof. Let $J = \{b(\ell), \ldots, \ell\} \cup \{b(q), \ldots, b(\ell)\}$ and $U = \bigcup_{j \in J} K_j \setminus (K_{b(\ell)} \cap K_\ell)$.
We show $N(v) \setminus U = K_{b(\ell)} \cap K_\ell$ for every $v \in U$. By definition, $K_{b(\ell)} \cap K_\ell \subseteq N(v)$.
Suppose for contradiction, $N(v) \setminus U \not\subseteq K_{b(\ell)} \cap K_\ell$, there exists $j \notin J$ such that $v \in K_j$, then $j < b(\ell)$ and there is $k < q$ such that $b(k) = j$.
Then v is at both sides of $K_{b(\ell)}$, and hence $v \in K_{b(\ell)}$; for the same reason, $v \in K_\ell$.
But then v is in $K_{b(\ell)} \cap K_\ell$, and should not be in U, a contradiction. □

Corollary. If an interval graph has only clique modules, then it has a unique clique path.
Theorem (Hsu 1995). A prime interval graph has a unique clique path.

The main lemma

Proposition.
Let π be a well-anchored ordering of an interval graph G, and $\pi^+ = \text{LBFS}^+(G, \pi)$. For any module M, $\pi|_M$ is a well-anchored ordering of $G[M]$; $\pi^+|_M = \text{LBFS}^+(G, \pi|_M)$.

Lemma (C 2021). Let π be a well-anchored ordering of an interval graph G, then π^+ is consistent with some clique path of G.

Proof sketch. Assume that G is connected and has no universal vertices.
A *major* module is a maximal non-clique module M with no universal vertex.
Major modules are pairwise disjoint.
Fix a clique path $\mathcal{K}$ of G, where for each major module M, the subpath of $\mathcal{K}$ for M is consistent with $\pi^+|_M$.
Suppose for contradiction that π^+ is not consistent with $\mathcal{K}$.
There are v_p, v_q s.t. $v_p <_\sigma v_q$ but $v_p >_\mathcal{K} v_q$. They aren't in the same major module.
We can find a module containing v_p and v_q. But this is impossible. $\square$

Generating a well-anchored ordering

Input: A graph G, and an $\textsc{LBFS}^+$ ordering τ^+ of G.
Output: A well-anchored ordering π of G if G is an interval graph.

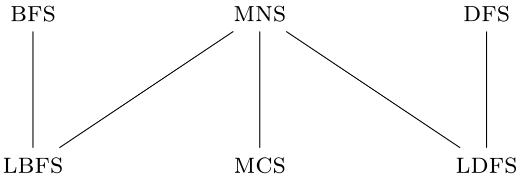
1. renumber the vertices such that $\tau^+(v_i) = i$ for all $i = 1, \ldots, n$;
2. **for** $i = 1, \ldots, n$ **do**
   - 2.1. $S \leftarrow$ unvisited vertices with the lexicographically largest label;
   - 2.2. $v_p \leftarrow$ the first vertex of $\tau^+|_S$;
   - 2.3. $v_q \leftarrow$ the last vertex of $\tau^+|_S$;
   - 2.4. **if** there exists $\ell < p$ such that $v_\ell \in N(v_p)$ and $\pi(v_\ell)$ is unset **then**
     $\pi(v_p) \leftarrow i$;
   - 2.5. **else if** there exist $v_\ell \in S$ and $v_r \in N(v_\ell)$ such that $r > q$ **then**
     $\pi(v_\ell) \leftarrow i$; (The main trick.)
   - 2.6. **else** $\pi(v_q) \leftarrow i$;
   - 2.7. **for each** unvisited neighbor of v **do**
     add i to $\mathrm{label}(v)$;
3. **return** π.

The algorithm of Li and Wu 2014

Input: A connected graph G.
Output: Whether G is an interval graph.

1. $\tau \leftarrow$ an LBFS ordering of G;
2. $\tau^{+} \leftarrow \text{LBFS}^{+}(G, \tau)$;
3. $\pi \leftarrow \text{LBFS}^{\uparrow}(G, \tau^{+})$;
4. $\pi^{+} \leftarrow \text{LBFS}^{+}(G, \pi)$;
5. **if** π^{+} is an interval ordering of G **then return** "yes";
6. **else return** "no."

Graph searches deserve more attention



A line between x (above) and y (below) indicates that y extends x.

- Corneil and Krueger 2008

  Lexicographic depth first search (LDFS)

  Maximal neighborhood search (MNS)
- Tarjan and Yannakakis 1984

  Maximal cardinality search (MCS)

Thanks!

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The algorithm of Hsu and Ma

Proposition. G is an interval graph if and only if
(1) $N(M)$ is a clique; and (2) both $G' = G - (M \setminus v)$ and $G[M]$ are interval graphs.

Find all the maximal modules, and the quotient graph, and work on them one by one.
The benefit is that the clique path is unique.

They developed the first linear-algorithm for modular decomposition of chordal graphs.
Any vertex v and a module M

- $v \in M$; the trouble case.
- $M \subseteq N(v)$; or
- $M \cap N(v) = \emptyset$.

It can be avoided if v is the vertex of the largest degree: It is only in trivial modules.

Proposition.
In a non-complete chordal graph, a maximum-degree vertex is only in clique modules.